Introduction

This thesis is dedicated to the numerical treatment of operator equations using adaptive quarklet methods. By a *quarklet* we understand the product of a wavelet and a piecewise polynomial, consequently we call a polynomially enriched wavelet basis *quarklet frame*. These function systems can initially be constructed both on the real line and the unit interval and, having done this, be generalised to higher dimensions by using tensor product techniques and domain decompositions. Quarklet systems are stable in function spaces, furthermore they fulfil certain compressibility properties and hence are convenient to be utilised in generic frame methods for the treatment of operator equations. Furthermore they represent a wavelet version of *hp*-methods, therefore there is strong hope that they converge quite fast.

Scope of Problems

Numerous problems in science and technology can be described with the help of *partial differential equations*. For example, the heat distribution in some material can be described as the solution of a partial differential equation, which is dependent on the time and the space coordinates. The heat equation belongs to the class of parabolic partial differential equations. A further class of time-dependent equations, e.g., the wave equation, are the hyperbolic differential equations. However, we restrict our discussion to the case of *elliptic* partial differential equations, which describe stationary phenomena, such as the deflection of a membrane or the bending of a board. To ensure unique solutions, one additionally has to incorporate boundary values. The most prominent example of an elliptic boundary value problem is the Poisson equation with homogeneous Dirichlet boundary conditions:

$$\begin{aligned} -\Delta u &= g \quad \text{in } D, \\ u &= 0 \quad \text{on } \partial D. \end{aligned}$$

Here, Δ denotes the Laplacian, g a continuous function, $D \subset \mathbb{R}^d$ a bounded, open, and connected set with boundary ∂D and u the unknown solution. Under certain additional assumptions elliptic boundary value problems belong to the class of elliptic operator equations in Hilbert spaces. In contrast to ordinary differential equations which just depend on one variable, partial differential equations are rather hard to solve analytically. Very often there does not exist a closed form of the solution. Furthermore, the concept of classical differentiable functions turns out to be too restrictive for certain boundary value problems. To overcome this obstacle, one treats a weaker concept of solutions. The boundary value problem is transformed into an operator equation on a *Sobolev space* $H_0^m(D)$ which is induced by a bilinear form $a(\cdot,\cdot)$:

$$a(u,v) = \langle g, v\rangle_{L_2} \quad \text{for all } v \in H_0^m(D).$$

Since the Sobolev spaces are infinite-dimensional generally, one can only gain an approximate solution by implementable methods.

Established Methods

Since this thesis is concerned with elliptic boundary value problems in their weak formulation, we exclude solution methods like finite difference methods that treat the classical formulation. The *Galerkin scheme* is a well-established method for the solution of elliptic boundary value problems. There the problem is considered on a finite-dimensional subspace of the solution space. This subspace is equipped with a basis $\{b_i\}_{i=1,\dots,N}$. With a bijection between $\mathbb{R}^d$ and this space, one can switch to a system of linear equations with coefficient matrix $(a(b_j, b_i))_{i=1,\dots,N, j=1,\dots,N}$. If the ansatz space in a certain sense is close to the solution space, then one can guarantee that the approximate solution u_N is close to the exact one:

$$\|u - u_N\|_{H_0^m(D)} \lesssim \inf_{w \in V_N} \|u - w\|_{H_0^m(D)}.$$

Here, V_N denotes the N-dimensional subspace and u_N the solution computed on this space. Prominent representatives of Galerkin schemes are the *finite element methods* (FEM). There local basis functions are constructed associated with a decomposition of the domain. A finer decomposition leads to a higher dimension of the ansatz space and a more precise solution. Hence finite element methods provide a geometric concept. The width of the mesh is usually denoted by h, therefore we denominate methods that are based on a solution space decomposition with respect to the space coordinates as h-methods. For second order elliptic partial differential equations, piecewise linear ansatz functions are an obvious choice. Finite element methods that rely on increasing the polynomial degree of the ansatz functions are known as p-FEM. Even a combination of both versions is possible. For special cases of so-called hp-FEM exponential convergence has been achieved.

Furthermore, *adaptivity* plays a key role. In uniform schemes the refinement strategy is equally performed for all ansatz functions. Therefore, degrees of freedom and hence computing capacity are wasted in regions of the domain where the solution is already well approximated. Adaptive schemes are self-controlled in the sense that the exactness of a calculated solution is estimated and the refinement is performed only in regions where the error is probably large. Meanwhile it is well-known that adaptive schemes are superior if the solution of the partial differential equation is contained in certain Besov spaces. In particular for h-FEM, there is a huge amount of literature, we refer to [17, 54, 76, 89] for an overview. For hp-FEM we refer to [8–12, 47, 62].

In contrast to the geometric concept of finite element methods, *wavelet methods* represent a basis orientated concept. Wavelet bases are certain bases for function spaces, in particular for Sobolev spaces. They provide strong analytical properties. It is possible to construct wavelets with arbitrarily high but finite regularity and still compact support. Furthermore they possess a certain amount of vanishing moments which lead to sparse representations of smooth functions. Wavelets have successfully been employed in techniques of signal analysis, but they can also be utilised for the discretisation of operator equations. In this case the resulting stiffness matrix is biinfinite and can be interpreted as an operator between sequence spaces. Certain properties of the wavelet basis and of the operator induce *compressibility* of the operator, therefore one can implement inexact versions of the theoretically feasible iterative solution methods. In the case of wavelet methods, the adaptivity reduces to approximately applying the stiffness matrix to some vector. Therefore, adaptive wavelet schemes are not faced with many typical problems that arise with adaptive finite element methods. We refer to [18–20, 81]. Usually wavelets are constructed by dyadic dilation and translation of a single function. Hence they belong to the class of h-methods. A classical wavelet basis Ψ of $L_2(\mathbb{R})$ has the form

$$\Psi := \left\{2^{j/2}\psi(2^j \cdot -k) : j, k \in \mathbb{Z}\right\}.$$

With certain additional assumptions, the classical wavelet characterisation of a Besov space reads as

$$\|f\|^q_{B^s_q(L_q(\mathbb{R}))} \sim \sum_{j\geq -1} \sum_{k\in\mathbb{Z}} 2^{j(s+\frac{1}{2}-\frac{1}{q})q} |\langle f, \psi_{j,k}\rangle_{L_2(\mathbb{R})}|^q.$$

Quarklet Schemes

A large part of this thesis will be concerned with the obvious question if it is possible to design wavelet versions of hp-methods for the solution of elliptic operator equations. This task can be decomposed into several smaller steps.

Construction of Univariate Quarklet Systems

A first step to establish adaptive quarklet methods is to construct stable function systems in the univariate setting. *Quarkonial decompositions* of function spaces have been invented by H. Triebel in [87]. In [30] the polynomial enrichment of partitions of unity has been discussed and their stability in Besov spaces was shown. Important tools are L_q stability and *Bernstein inequalities* of the enriched functions. Further assumptions on the underlying system were formulated, e.g., *Jackson inequalities*. In loc. cit. there also was presented a proof technique of successively considering higher enriched systems which can be transferred to the case of an underlying wavelet basis.

In [68] quarklets were primarily mentioned as Haar wavelets which are enriched by orthogonal polynomials. In [28] quarklet frames on the real line based on the CDF wavelet basis have been discussed. In particular their compressibility and applicability for adaptive frame schemes were proven. Finally, in [26] quarklet frames on the interval have been constructed, where the major difference to the shift-invariant case lies in the appropriate construction of boundary functions. We briefly describe the construction of univariate quarklets. It is based on an underlying biorthogonal spline wavelet basis. This wavelet basis has been constructed with the help of a multiresolution analysis, that means the wavelets are linear compositions of generators. In our particular case those generators are B-splines. Then, a *quark* φ_p is defined as the product of a monic polynomial and a B-spline φ:

$$\varphi_p := \left(\frac{\cdot}{\lceil \frac{m}{2} \rceil}\right)^p \varphi, \quad p \in \mathbb{N}_0.$$

Then, the quarklet ψ_p is defined by

$$\psi_p := \sum_{k \in \mathbb{Z}} b_k \varphi_p(2 \cdot - k),$$

where $b = \{b_k\}_{k \in \mathbb{Z}}$ denotes the original wavelet mask. Since the quarklets are linear combinations of quarks, it suffices to primarily study the properties of the quarks. One of these properties is refinability in the sense that vectors of quarks are refinable. In [31] it has been shown how these multiquarks fit in the framework of more general multigenerators and multiwavelets. Further important properties are, as mentioned above, stability properties of the single-scale functions, Bernstein inequalities and vanishing moments. These properties allow to deduce equivalent norms for Besov spaces. These results are stated in Theorem 4.27 and 4.28. With suitable weights the frame property in L_2-Sobolev spaces immediately follows:

$$\|f\|_{H^m}^2 \sim \sum_{\lambda \in I} |\langle f, w_\lambda \psi_\lambda \rangle_{H^m}|^2.$$

Quarklet Frames on Domains

A large class of elliptic operator equations is considered on domains which are decomposable into diffeomorphic transformed unit cubes. We restrict our discussion to the case of translated cubes, for more general domains we refer to [39] . In a first step of generalisation one has to construct quarklet frames on cubes. A tensor product approach is favourable since the Sobolev spaces in two dimensions provide the following structure which can be generalised to higher dimensions:

$$H^m([0,1]^2) = H^m([0,1]) \otimes L_2([0,1]) \cap L_2([0,1]) \otimes H^m([0,1]).$$

By performing the following two steps one can even construct more general tensor product frames on cubes. At first, tensor products of frames are frames for tensor products of Hilbert spaces if the weights are appropriately chosen. Subsequently, rescaled frames which contain a Riesz basis are frames for intersections of Hilbert spaces. Afterwards, domains like the L-shaped domain, which represents a prominent test case for adaptive algorithms, are decomposed into cubes in a non-overlapping way. Based upon a quarklet frame on each cube a quarklet frame on the whole domain is constructed. We proceed similar to the wavelet case described in [13]. Initially, one has to choose boundary conditions for the cubes. Then, quarklets defined on a cube are suitably extended to the neighbouring cubes. For second order differential equations the simple reflection turns out to be a suitable extension operator. The main result of this chapter is presented in Theorem 5.15.

Compressibility

Having constructed frames of quarklet type, they are ready to be utilised in generic frame methods if they possess certain compression properties. That means, the biinfinite stiffness matrix $\boldsymbol{A}$ has to be well approximated by finite matrices $\boldsymbol{A}_J$:

$$\sum_{J\in\mathbb{N}} 2^{Js}\|\boldsymbol{A}-\boldsymbol{A}_J\| < \infty \quad \text{for all } 0 \le s \le s^*.$$

In [28] compressibility for the univariate Laplacian has been shown, in [26] this result has been generalised to higher dimensions. The Laplacian is an important special case, nevertheless the compression results can be transferred to other elliptic operators. The compressibility heavily relies on the vanishing moments property of the quarklets and affects the performance of adaptive quarklet schemes. To achieve faster convergence one can distinguish between *first* and *second compression*. Second compression in the case of wavelets has been studied in [34, 80]. One can use the fact that the quarklets possess a higher local regularity in the design of more involved compression strategies to gain better results. The univariate main result of this Section is presented in Theorem 6.7. The different strategies and results in the multivariate case are described in Theorems 6.15, 6.17, 6.18.

Approximation of Singular Solutions

It is of course a long way to go to achieve provable exponential convergence of adaptive quarklet schemes, though a necessary condition is that typical solutions to partial differential equations can be well approximated in terms of quarklets. It is well known that domains with re-entrant corners, such as the L-shaped domain, induce singular solutions. The univariate function x^α with $\alpha > \frac{1}{2}$ serves as a model for such solutions. We show that this particular function can be approximated in terms of quarklets such that the error decreases exponentially with respect to degrees of freedom. To prove this, similar to [2], a certain spline with varying polynomial degree is constructed on a partition of the interval $[0, 1]$ that geometrically becomes finer advancing to the left boundary. In contrast to that result, the spline described here is $m-2$ times continuously differentiable. The construction can be found in Theorem 7.1. This particular spline can be developed into single-scale quarks and, with the help of reconstruction properties, transformed into quarklet frame elements. To perform these transformations, we consider two different methods, firstly a general reconstruction principle that fits in the framework of general multiwavelets, secondly an adapted reconstruction principle that even provides a better asymptotic error decay. The results for the cases $L_2(I)$, $H^1(I)$ and $H^1(I^2)$ are presented in the Theorems 7.9, 7.11, 7.13, respectively.

Reconstruction of Multigenerators

Vectors of quarks or quarklets fit in the framework of multigenerators and multiwavelets, respectively. This general concept is, e.g., applied to the construction of orthonormal interpolating wavelets which can not be realised with a single generator. We use Fourier techniques to derive a general criterion for the reconstruction of multigenerators in Theorem 3.2 and apply this result to the special case of multiquarks in Theorem 4.15.

Layout

Parts of this thesis have been published in [26, 27, 31]. We proceed in the following way. In Chapter 1 we treat the necessary foundations. In Section 1.1 we introduce function spaces, in particular Sobolev and Besov spaces. In Section 1.2 we recall some basic facts about elliptic partial differential equations and their weak formulation. Section 1.3 is dedicated to frames. In Chapter 2 we recall the construction of biorthogonal wavelet bases. Firstly we consider the classical construction of a wavelet basis of $L_2(\mathbb{R})$, secondly we treat the construction of a wavelet basis on the interval. Two particular spline wavelet bases are of our interest, namely the CDF wavelet basis from [21] and the basis constructed by M. Primbs in [66]. There reconstruction properties appear as a by-product. In Chapter 3 we derive a general criterion for multigenerators to fulfil certain reconstruction properties. In Chapter 4 we introduce univariate quarklets. In Section 4.1 we recall the construction and properties of quark generators on the real line and the interval, respectively. In Section 4.2 we apply the results from Chapter 3 to derive reconstruction properties of the shift-invariant quarklets. In Section 4.3 we adapt quarklets to the interval. This is done by the appropriate construction of boundary functions. The remainder of this chapter is dedicated to the stability of quarklet systems in function spaces, in particular in Section 4.5 and 4.6 we derive equivalent norms for Besov spaces. The known special cases of L_2 and L_2-Sobolev spaces are recalled in Section 4.4 and 4.7, respectively. Having shown the frame property of quarklet systems, it is possible to design adaptive quarklet schemes. This is done in Chapter 6. In Sections 6.2, 6.3 the crucial compression properties of quarklet frames are treated, where we pay attention to second compression strategies. Finally, in Chapter 7 we study approximation properties of the quarklets in L_2 and H^1 norms, respectively. Using the tensor product ansatz from Section 5.1, it suffices to consider univariate quarklet approximation for certain singular solutions to elliptic boundary value problems. The thesis closes with several numerical experiments in Chapter 8. As a canonical example we treat the Poisson equation on two-dimensional domains with re-entrant corners numerically. Furthermore, second compression and quarklet approximation are tested.